
ResonanT⁴ Foundation

Modular Closure and Transverse Geometry

Egils Jakovels

Independent researcher

The author developed this work through iterative conceptual exploration, formalization, and critical refinement. AI-assisted tools were used for reasoning support and editorial refinement. All conceptual direction, theoretical decisions, and responsibility for the content rest solely with the author.

Version 1.1.0 · DRAFT · DOI: [10.5281/zenodo.18157442](https://doi.org/10.5281/zenodo.18157442) (resolves to v1.0.0; this revision is not yet deposited)

Abstract. ResonanT⁴ is a structure-first framework in which admissible configurations are defined by exact phase closure. Intrinsic structure consists of finite modular orbits with rational increments, ensuring boundedness by arithmetic necessity. Apparent motion arises from transverse projection and does not represent dynamical time evolution.

Contents

Abstract

1. Ambient Configuration Space and Admissibility
 - 1.1 Scope: admissibility is structural, not a persistence claim
2. Objects, Equivalence, and Invariants
3. Intrinsic Modular Construction
 - 3.1 Intrinsic Phases
 - 3.2 Global Phase Offset
4. Modular Orbit and Closure
5. Locking Structure
6. Cyclic Deviation and Approximate Closure
 - 6.1 Two independent gradings
7. Factorization Freedom
8. Transverse Phase and Projection
9. Canonical Embedding (Example)
10. Conclusion

Revision history

References

Abstract

ResonanT⁴ is a structure-first modular framework in which **admissibility is defined by exact phase closure**. The theory studies a restricted class of configurations – finite modular orbits with rational increments – embedded within a broader ambient space of possible phase relations. Closure is not treated as a dynamical outcome but as a structural constraint selecting bounded, repeatable configurations. Apparent motion arises solely from transverse projection and does not represent physical time evolution.

1. Ambient Configuration Space and Admissibility

Consider a general configuration space $\mathcal{C}$ of phase increment sets, including continuous, irrational, or perturbed relations. Most elements of $\mathcal{C}$ generate non-repeating, unbounded trajectories.

ResonanT⁴ restricts attention to a **closure-compatible subset** $\mathcal{A} \subset \mathcal{C}$, defined by exact modular closure. This restriction is intentional: the framework characterizes the internal structure of admissible configurations.

1.1 Scope: admissibility is structural, not a persistence claim

Admissibility as defined here is a property of a configuration: whether its phase increments return exactly. It is not, by itself, the assertion that admissible configurations are the ones that survive perturbation. Those two statements come apart, and the direction in which they come apart depends on the class of evolution applied.

- Under **conservative (Hamiltonian) perturbation**, exactly closing configurations are the fragile ones. This is the content of KAM theory: resonant tori are destroyed first, and survival requires a Diophantine condition, which is precisely the statement that the closure deficit is bounded away from zero [1,2]. The last invariant circle to break is the one whose rotation number is worst approximable by rationals [3,4].
- Under **driven dissipative evolution**, exactly closing configurations are the persistent ones. Rational rotation numbers occupy locking regions of finite width in parameter space and attract a neighborhood of initial conditions [5].

The same quantity therefore carries opposite preference in the two regimes. Any paper built on this foundation must state which regime it operates in before converting admissibility into a claim about what persists; the companion papers do so explicitly, on the driven dissipative

side in [6] and on the conservative side in [7]. The present paper makes no such conversion: it is non-dynamical throughout, and the classification below is a statement about structure only.

2. Objects, Equivalence, and Invariants

The primary object studied in this paper is an **intrinsic modular orbit** defined by integer parameters

(n, k, ξ, ω) and the induced phase map

$$u \mapsto (\Theta(u), \Xi(u), \Phi(u)), \quad u \in \mathbb{Z}_n.$$

Candidate space.

The ambient candidate space $\mathcal{C}$ consists of all phase increment triples $(\Delta\Theta, \Delta\Xi, \Delta\Phi) \in \mathbb{R}^3$, including irrational and perturbed relations.

Admissible subset.

The admissible subset $\mathcal{A} \subset \mathcal{C}$ consists of those increments satisfying exact modular closure after a finite number of steps.

Equivalence relation.

Two admissible configurations are equivalent if they generate the same intrinsic modular orbit up to cyclic relabeling. Equivalently, they share the same integer invariants (n, k, ξ, ω) modulo overall gcd reduction.

Invariants.

The following quantities are invariant under admissible deformations:

- the gcd $g = \gcd(n, k, \xi, \omega)$,
- the number of locked strands g ,
- the primitive cycle length n/g ,
- the winding integers $(k/g, \xi/g, \omega/g)$.

Exact vs approximate closure.

Exact closure defines discrete equivalence classes. Near-closure configurations form neighborhoods around these classes, as quantified later by cyclic deviation. These neighborhoods underpin the approximate closure basins used in Paper II [6].

3. Intrinsic Modular Construction

Let $n, k, \xi, \omega \in \mathbb{Z}$. Define the index

$$u = \frac{t}{n}, \quad t \in \mathbb{Z}_n.$$

The index labels a finite cyclic set and does **not** represent time.

3.1 Intrinsic Phases

$$\Theta(u) = 2\pi k u + \phi_0, \quad \Xi(u) = 2\pi \xi u, \quad \Phi(u) = 2\pi \omega u + \psi.$$

The triple (Θ, Ξ, Φ) defines an intrinsic phase state. Closure follows arithmetically from integer increments.

3.2 Global Phase Offset

The constant ϕ_0 fixes absolute orientation but does not affect closure, locking, or admissibility.

4. Modular Orbit and Closure

The modular increment is

$$\Delta = \left(\frac{2\pi k}{n}, \frac{2\pi \xi}{n}, \frac{2\pi \omega}{n} \right).$$

The intrinsic modular orbit

$$\mathcal{O} = \{(\Theta(u), \Xi(u), \Phi(u)) : t \in \mathbb{Z}_n\}$$

is finite. The primitive closure period is

$$n/g, \quad g = \gcd(n, k, \xi, \omega).$$

Proof sketch.

Closure requires the smallest positive integer m such that $m(k, \xi, \omega) \equiv (0, 0, 0) \pmod{n}$. Writing $n = gn'$ and $(k, \xi, \omega) = g(k', \xi', \omega')$ with $\gcd(n', k', \xi', \omega') = 1$, the minimal solution is $m = n'$, yielding a primitive period n/g .

5. Locking Structure

If $g = 1$, the orbit forms a single cycle. If $g > 1$, it decomposes into g disjoint phase-locked cycles of equal length. Locking structure is an invariant of the closure class.

Proof sketch.

When $g > 1$, the index set $\mathbb{Z}_n$ partitions into g cosets modulo n/g . Each coset advances phases by identical increments modulo 2π , producing g independent cycles of length n/g .

6. Cyclic Deviation and Approximate Closure

Decompose

$$n = qk + \delta, \quad 0 \leq \delta < k.$$

Define the cyclic deviation

$$\delta_* = \min(\delta, k - \delta).$$

The quantity δ_* measures distance from exact commensurability. Configurations with small δ_* form neighborhoods in configuration space where closure deficit remains bounded.

This allows a graded notion of admissibility: exact closure defines the core class, while near-closure configurations define structured neighborhoods relevant for perturbative or numerical contexts.

6.1 Two independent gradings

Admissibility as used across this framework has two distinct aspects, and each is graded by a different quantity:

- **Whether a configuration closes.** The closure deficit C measures distance to exact closure. It separates near-closure from far-from-closure and is the appropriate grading on the detuned or irrational side.
- **How cheaply it closes.** The primitive period n/g measures the index length required to return. It separates a short cycle from a long one.

These gradings are independent, and **no single scalar carries both**. For any denominator budget $N \geq n/g$, every exactly closing configuration satisfies $C=0$, so the deficit assigns them all the same value while their primitive periods differ without bound. Conversely, the primitive period is undefined off the closure set.

The distinction has consequences wherever admissible configurations are ranked against one another. A construction that ranks near-closure neighborhoods must use C ; one that ranks exactly closing configurations must use n/g . In the driven dissipative setting the relevant scaling is by primitive period, since locking-region width falls off geometrically with the period of the rational being locked to [5]. A single admissibility score built from C alone cannot reproduce that ordering, because it is constant on the entire set being ordered.

7. Factorization Freedom

Closure determines a finite index set $\mathbb{Z}_n$ but does not uniquely specify connectivity.

A connection generator $s \in \mathbb{Z}$ induces

$$t \sim t + s \pmod{n},$$

yielding distinct cycle factorizations without altering admissibility.

8. Transverse Phase and Projection

Introduce a transverse geometric parameter τ . The parameter τ is **not time** and does not modify intrinsic phase relations.

Variation of τ generates equivalent spatial projections of the same closure-stable structure. Apparent motion is purely geometric and observer-dependent.

9. Canonical Embedding (Example)

A representative embedding

$$F : (\Theta, \Xi, \Phi, \tau) \mapsto (x, y, z) \in \mathbb{R}^3$$

may be defined for visualization. The choice of embedding is non-unique.

10. Conclusion

ResonanT⁴ establishes closure as a structural admissibility criterion and provides a complete internal classification of closure-compatible configurations. The framework is intentionally non-dynamical and serves as a foundation for further theoretical or applied extensions.

Two boundaries are stated explicitly rather than left implicit. Admissibility is structural and does not by itself determine what survives perturbation (§1.1); and admissibility is graded by two independent quantities, the closure deficit and the primitive period, which no single scalar combines (§6.1). Extensions that convert this classification into claims about persistence or control inherit both boundaries.

Revision history

v1.1.0. Scope clarifications following internal review. No definition, classification, or result in this paper is withdrawn or altered.

- Added §1.1. Admissibility is a structural property and does not by itself determine what survives perturbation. The relationship between exact closure and persistence carries opposite signs under conservative and under driven dissipative evolution, and papers building on this foundation must state their regime before converting admissibility into a persistence claim.
- Added §6.1. Admissibility is graded by two independent quantities, the closure deficit and the primitive period, and no single scalar carries both.
- Added a reference list, previously absent, and citations for the results and companion papers the text relies on.
- Typographic fix in the abstract: two parenthetical dashes had been flattened to hyphens, joining words as "configurations-finite" and "increments-embedded". Wording is unchanged.

These additions state boundaries that were implicit in v1.0.0 and that the companion papers had inherited without qualification.

v1.0.0. Initial preprint.

References

1. Kolmogorov, A. N., On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian. *Dokl. Akad. Nauk SSSR* **98**, 527–530 (1954).
2. Arnold, V. I., Proof of a theorem of A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian [<https://doi.org/10.1070/RM1963v018n05ABEH004130>]. *Russian Math. Surveys* **18**, 9–36 (1963).
3. Greene, J. M., A method for determining a stochastic transition [<https://doi.org/10.1063/1.524170>]. *J. Math. Phys.* **20**, 1183–1201 (1979).
4. MacKay, R. S., A renormalisation approach to invariant circles in area-preserving maps [[https://doi.org/10.1016/0167-2789\(83\)90129-6](https://doi.org/10.1016/0167-2789(83)90129-6)]. *Physica D* **7**, 283–300 (1983).
5. Arnold, V. I., Small denominators I: mappings of the circumference onto itself. *Amer. Math. Soc. Transl.* **46**, 213–284 (1965).
6. Jakovels, E., Closure-Compatible Resonance, Admissibility Gradients, and Phase-Aligned Control [<https://resonant4.io/papers/closure-compatible-resonance>]. *resonant4.io* (2026). doi:[10.5281/zenodo.18157799](https://doi.org/10.5281/zenodo.18157799) [<https://doi.org/10.5281/zenodo.18157799>].

7. Jakovels, E., Quantum Numbers as Closure Invariants [<https://resonant4.io/papers/quantum-closure-invariants>]. *resonant4.io* (2026).
doi:[10.5281/zenodo.18157869](https://doi.org/10.5281/zenodo.18157869) [<https://doi.org/10.5281/zenodo.18157869>].